STABILITY OF A HOLLOW VORTEX

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There are a large number of theoretical and experimental works concerned with the study of the stability of a rotating Couette flow [1-3]. The emphasis is on axisymmetric perturbations as being the most critical.

The results of experiments and calculations show that when cylinders are rotating in the same direction, as the relative rotational velocity of the outer cylinder increases, the critical Taylor number, for which the starting laminar regime is destroyed and Taylor vortices arise, increases.

For $\omega_1 r_1^2 \approx \omega_0 r_0^2$, the graph of the function $T_*(\omega_2 r_1^2/\omega_0 r_0^2)$ (ω_0, ω_1 are the angular rotational velocity of the inner and outer cylinders, r_0 and r_1 are the radii of the corresponding cylinders) becomes asymptotic.

The flow of a rotating fluid in the presence of an infinite rigid wall for one of the surfaces has been less studied. Such flows often appear in the chemical and paper industries with cooling of rotating mechanisms by liquid films, and so on. The appearance of perturbations in the liquid leads to a distortion of the free surface and the appearance of waves on it, and this affects the general picture of stability.

The problem of the stability of a rotating flow, when one of the boundaries is free, while the other boundary is rigid or at infinity, is examined in [4]. The stability relative to small nonviscous perturbations is examined.

An experimental investigation of the stability of a flow with a free outer surface and a numerical analysis are carried out in [5]. A change in the surface by wave formations is observed for a certain ratio of the angular velocity (rigid body rotation is examined) and the thickness of the film.

The form of the equilibrium of the fluid located on the outer or inner surfaces of the solid boundary and rotating together with it as a solid body is examined in [6]. Taking into account surface tension, it was found that in the perturbed motion the velocity field remains the initial field, while the free boundary is perturbed.

The effect of the deformation of the free surface on the convective instability of the horizontal layer was studied in [7]. The deformability of the surface leads to a decrease in stability, and this effect is greatest in the absence of surface tension [8].

In the present work, a numerical analysis of the linear stability of a rotating flow bounded by an outer cylindrical wall and having an inner free surface with radius r_0 is analyzed. Three-dimensional perturbations are examined. Surface tension is neglected.

With the flow of a dripping liquid in a vortical cell with a tangential inlet, a non-flow-through zone arises in the region of the cell near the axis. This occurs as a result of the stratification of the fluid with circulation [9]. In [10], it is shown that the flow in the vortical cells being examined has the properties required for creating a non-flow-through region with a cylindrical or torroidal shape.

The study of the stability of the developing flows with inner boundaries is a very difficult problem. However, it is possible to examine the problem approximately, isolating for the analysis certain regions of the flow. In examining the stability of the flow established, it is possible to fix the inner boundary or give a stationary velocity field and study its stability over the entire cross section.

Qualitative information can be obtained by studying the simplest one-dimensional problem of the stability of a hollow vortex.

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1. Let small perturbations of the form

$$w_* = w(r) \exp[i\alpha(z - ct) + im\varphi], \quad w(r) = \{w_r, w_\varphi, w_z\},$$
$$p_* = -(i/\alpha \operatorname{Re}) q \exp[i\alpha(z - ct) + im\varphi]$$

be superimposed on the main flow with vector velocity $V_0 = \{0, v(r), u(r)\}$ and pressure p_0.

The corresponding equations for the perturbations in a cylindrical system of coordinates r, φ, z are obtained in [11]

$$\begin{gathered}
aw_r - \frac{2\operatorname{Re} v}{r} w_\varphi = \frac{i}{\alpha} q' + \left\{\frac{1}{r}(rw_r)'\right\}' - \frac{2mi}{r^2} w_\varphi, \\
aw_\varphi - \operatorname{Re}\left(v' + \frac{v}{r}\right) w_r = -\frac{m}{\alpha r} q - \left\{\frac{1}{r}(rw_\varphi)'\right\}' + \frac{2mi}{r^2} w_r, \\
aw_z + \operatorname{Re} u' w_r = -q + \frac{1}{r}(rw_z')', \\
(rw_r)' + i\alpha r w_z + imw_\varphi = 0, \quad a = i\alpha \operatorname{Re}(u - c) + \frac{im\operatorname{Re}}{r} v + \frac{m^2}{r^2} + \alpha^2.
\end{gathered} \tag{1.1}$$

where w_r, w_φ, and w_z are the components of the amplitude perturbations of the velocity; α is the axial wave number, which can vary from zero to ∞; m = 0, 1, 2,... is the azimuthal wave number; c = X + iY is the sought-after characteristic value of the problem (1.1) (for Y < 0, the perturbations are damped in time); v(r) and u(r) are the azimuthal and axial components of the unperturbed velocity; Re is Reynolds number. The prime denotes differentiation with respect to r.

The length scale is taken as the magnitude of the gap between the surfaces bounding the flow and, in this case, the dimensionless radius of the inner boundary ξ will be a parameter determining the geometry of the channel. For $\xi \gg 1$, the gap becomes small; for $\xi \to \infty$ in the limit, we have a flat channel; for $\xi \to 0$, a pure tube. The instantaneous dimensionless radius r varies in the interval [ξ, ξ + 1].

The velocity scale is the magnitude of the axial component, averaged over the transverse cross section of the channel. The Reynolds number is constructed according to the chosen scales for length, velocity, and kinematic viscosity ν.

Another characteristic parameter, the Taylor number, which can be represented in the form $\sqrt{T} = \sqrt{2/(1+2\xi)}\, v_0 \operatorname{Re}$, where v_0 is the magnitude of the dimensionless azimuthal velocity of the inner boundary, is introduced.

The boundary conditions on the outer cylinder follow from the conditions of sticking and impermeability

$$w_r(1+\xi) = w_\varphi(1+\xi) = w_z(1+\xi) = 0. \tag{1.2}$$

On the inner boundary, the tangential stresses are maintained at such a value that the axial component of the velocity equals u_0, while the azimuthal component is v_0. The normal stress is constant. We will obtain the boundary conditions on the inner surface taking into account its deformability.

Let a perturbation of the form

$$r_* = \varepsilon(r) \exp[i\alpha(z - ct) + im\varphi]$$

be imposed on the inner boundary.

Then, the tangential and normal stresses on the inner surfaces $r = \xi + r_*$ are expressed in the form

$$-\frac{i}{\alpha \operatorname{Re}} q + \frac{2}{\operatorname{Re}} w_r' - \frac{im}{r\operatorname{Re}} \varepsilon\left(v_0' - \frac{v_0}{r}\right) - \frac{i\alpha\varepsilon}{\operatorname{Re}} u_0' - \frac{v_0^2}{r}\varepsilon = 0,$$
$$w_\varphi' - \frac{w_\varphi}{r} + \frac{imw_r}{r} + \varepsilon\left(v_0' - \frac{v_0}{r}\right)' = 0, \quad w_z + i\alpha w_r + u_0''\varepsilon = 0.$$

Taking into account the impenetrability of the inner boundary and discarding second order terms, we obtain one more condition on the inner boundary:

$$\varepsilon(i\alpha c - imv_0/r - i\alpha u_0) + w_r = 0.$$

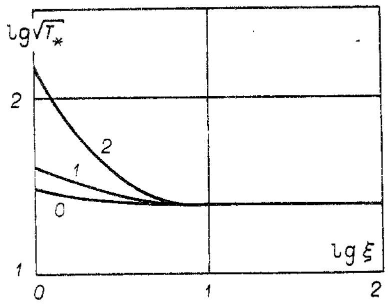

Fig. 1

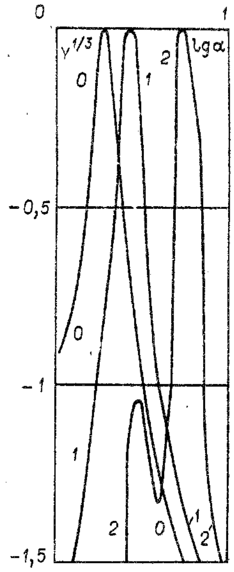

Fig. 2

We will eliminate the amplitude of the perturbation on the inner boundary and obtain the following boundary condition at $r = \xi$:

$$-\frac{i}{\alpha}q + 2w_r' + \left[\frac{im}{r}\left(v_0' - \frac{v_0}{r}\right) + i\alpha u_0' + \frac{\operatorname{Re} v_0^2}{r}\right]\frac{iw_r}{\alpha u_0 + \frac{mv_0}{r} - \alpha c} = 0,$$

$$w_\varphi' - \frac{w_\varphi}{r} + \frac{imw_r}{r} - \frac{iw_r}{\alpha u_0 + \frac{mv_0}{r} - \alpha c}\left(v_0' - \frac{v_0}{r}\right)' = 0, \tag{1.3}$$

$$w_z + i\alpha w_r - \frac{iw_r u_0''}{\alpha u_0 + \frac{mv_0}{r} - \alpha c} = 0.$$

The problem (1.1)-(1.3) is solved by the differential passage method with joining [11].

The velocity field of the laminar flow being studied is one-dimensional and depends only on the coordinate r. For impenetrable surfaces, the stationary solution of the Navier-Stokes equations has the form

$$v(r) \equiv 0,\ v_z = u(r) = Ar^2 - B\ln r + C,\quad v_\varphi = v(r) = E/r + D. \tag{1.4}$$

The expressions for the constants are obtained by satisfying the boundary conditions ($u = 0$, $v = v_1$ at $r = 1 + \xi$, $u = u_0$, $v = v_0$ at $r = \xi$) and taking into account the normalization condition

$$2\int_{\xi}^{\xi+1} ur\,dr = 1 + 2\xi:$$

$$B = \frac{(2 + u_0)(1 + 2\xi)}{1 + 2\xi - (1 + 2\xi + 2\xi^2)\ln\frac{1+\xi}{\xi}},\quad A = B\ln\left(\frac{1+\xi}{\xi} - u_0\right)\Big/(1 + 2\xi),$$

$$C = B\ln(1 + \xi) - A(1 + \xi)^2,\ D = [v_1(1 + \xi) - v_0\xi]/(1 + 2\xi),$$

$$E = v_1(1 + \xi) - D(1 + \xi)^2.$$

The azimuthal velocity distribution is not a rigid body distribution, and the cases for which $v_1(1 + \xi) < v_0\xi$ are examined.

2. The spectrum of characteristic values of the problem (1.1)-(1.4) with an inner boundary is calculated with the use of the method of motion along the discontinuity [12].

The stability of the flow between coaxial cylinders is examined in [13]. Using data from [13] and varying the boundary conditions continuously at $r = \xi$, we pass from the conditions of sticking and impenetrability on the inner cylinder to the given boundary conditions on the free surface (1.3) and obtain the spectrum of characteristic values of the problem (1.1)-(1.4) for arbitrary parameters ξ, Re, T, α, v_1, and v_0.

The critical Taylor numbers are obtained in [13] as a function of the Reynolds number for a spiral flow between coaxial cylinders, if the outer cylinder is stationary.

We will determine the effect of the free boundary on the stability picture of the flow. The calculations are carried out for small axial Reynolds numbers (Re = 1). Further decrease of Re does not effect the magnitude of the critical Taylor number, i.e., v_0 Re remains constant.

Numerical calculations showed that the free inner boundary leads to a decrease in the critical Reynolds number, i.e., the initial laminar regime is destroyed for the smallest relative rotational velocities of the inner boundary.

Taylor's number for a narrow gap for a flow between coaxial cylinders equals $\sqrt{T_*} = 41.3$ (ξ = 50, m = 0, Re = 1), while at the free boundary $\sqrt{T_*} = 23.8$ ($\xi = 50, m = 0$, Re = 1).

Decreasing ξ (increasing the gap) leads to an increase in the critical Taylor numbers for axisymmetric perturbations, as well as for three-dimensional perturbations with m = 1 and m = 2.

Figure 1 shows $\sqrt{T_*}$ as a function of ξ. The curves 0-2 correspond to a change in $\sqrt{T_*}(\xi)$ for perturbations with m = 0, m = 1, and m = 2. For ξ = 1, when the radius of the free surface is equal to one-half the radius of the outer cylinder, the critical Taylor numbers take on the following values: $\sqrt{T_*} = 29.6\,(m = 0)$, $\sqrt{T_*} = 38\,(m = 1)$, and $\sqrt{T_*} = 145$ (m = 2). As can be seen from Fig. 1, $\sqrt{T_*}$ for axisymmetric perturbations are smallest over the entire range of variation in ξ. A similar picture is also observed for spiral flow between coaxial cylinders with small Reynolds numbers:

$$\sqrt{T_*} = 56.5\,(m = 0), \quad \sqrt{T_*} = 62.6\,(m = 1), \quad \sqrt{T_*} = 77\,(m = 2).$$

In this case as well, the critical Taylor numbers of axisymmetric perturbations are smallest, i.e., the most critical perturbations are perturbations with m = 0. The critical wave numbers α_*, when ξ is decreased, remain constant ($\alpha_* \simeq 2$) for axisymmetric perturbations. If $m \neq 0$, α_* increases with an increase in the gap between the surfaces:

$$\alpha_* \simeq 2.6 \text{ for } m = 1,\ \xi = 1, \quad \alpha_* = 5.85 \text{ for } m = 2,\ \xi = 1.$$

Figure 2 shows Y as a function of the wave number α with the critical Taylor number corresponding to each azimuthal number m. The dependence of the decrement of the perturbations on the wave number is given for ξ = 1. The curves 0-2 characterize the changes in Y(α) for perturbations with m = 0, m = 1, and m = 2. It is evident from Fig. 2 that the function Y(α) for m = 0 has a single maximum at α = 2, while Y(α) for m = 1 has a maximum at α = 2.6.

The curve corresponding to the change in Y(α) for perturbations with m = 2 has two maxima at α = 3.16 and 5.85. If Taylor's number is increased, then the first maximum on curve 2 at α = 3.16 shifts toward longer wavelengths, remaining negative at the same time. The second maximum on curve 2 (α = 5.85) moves into the region of positive values and is responsible for the instability of the azimuthal mode with m = 2.

For all types of perturbations examined (m = 0, m = 1, and m = 2), α increases with an increase in the Taylor number.

Let us estimate the effect of the rotation of the outer cylinder on the stability of the flow. We will examine the case ξ = 50, when the gap is small, and ξ = 1, when the radius of the inner surface is one-half the outer radius.

The results of the calculations show that small relative velocities of rotation of the outer cylinder do not have a significant effect on the magnitude of the critical Taylor number and the wave numbers.

Let us introduce the dimensionless complex $v_1(1 + \xi)/v_0\xi$ and generalize the results of the calculations for ξ = 50 and ξ = 1.

A weak change in T_* and α_* is observed up to $v_1(1 + \xi)/v_0\xi \sim 0.5$.

Further increase in the parameter $v_1(1 + \xi)/v_0\xi$ leads to an increase in α_* for three-dimensional perturbations (m = 1, m = 2), and for axisymmetric perturbations (m = 0) α_* remains equal to 2.

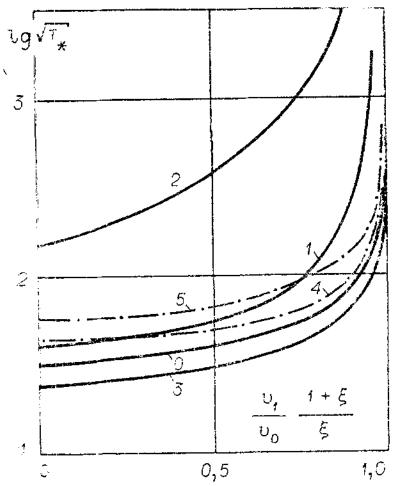

Fig. 3

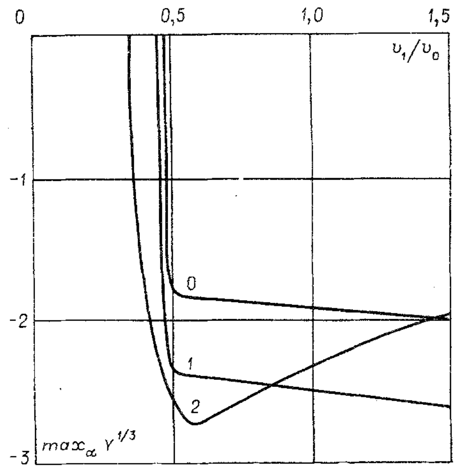

Fig. 4

The functions $Y(\alpha)$ have the same character as for $v_1(1 + \xi)/v_0\xi = 0$, i.e., in the absence of rotation of the outer cylinder, with the only difference that for perturbations with m = 1 and m = 2 the maxima on the $Y(\alpha)$ curves are shifted toward the short wavelength region.

The critical Taylor numbers increase with an increase in the relative rotational velocity of the outer cylinder for perturbations with m = 0, m = 1, and m = 2.

Figure 3 shows the change in $\sqrt{T_*}$ as a function of the parameter $v_1(1 + \xi)/v_0\xi$. The behavior of the curves 0–2 corresponds to the change in the critical Taylor numbers $\sqrt{T_*}$ $([v_1(1+\xi)/v_0\xi])$ for perturbations with m = 0, m = 1, and m = 2 in the case of a wide gap ($\xi = 1$). Curve 3 reflects the change in $\sqrt{T_*}$, if the gap is small ($\xi = 50$, m = 0).

The nature of the curves in Fig. 3 suggest that the axisymmetric perturbations m = 0 are the most critical over the entire range of variation of the parameter $v_1(1 + \xi)/v_0\xi$. These perturbations are localized near the free surface, and the phase velocities are of the order of the velocity of the axial flow. The critical wave numbers α_* remain constant along the curve $0(\alpha_* \simeq 2)$.

Perturbations with azimuthal number m = 2 are also localized near the free surface, i.e., they are "near-wall" perturbations, while perturbations with m = 1 have a near-axial character and are localized at the center of the channel. Curves 0–3 (Fig. 3) become asymptotic as the parameter $v_1(1 + \xi)/v_0\xi$ approaches unity.

For comparison, Fig. 3 shows the results of numerical calculations of the critical Taylor numbers as a function of the relative rotational velocities of the outer and inner boundaries for the flow between coaxial cylinders. Curve 4 reflects the change in $\sqrt{T_*}$ with increasing parameter $v_1(1 + \xi)/v_0\xi$ in the case of a narrow gap ($\xi = 50$). Along the curve 5, $\sqrt{T_*}$ changes for axisymmetric perturbations m = 0, when the gap between the cylinders equals the radius of the inner cylinder ($\xi = 1$).

Perturbations responsible for the instability of the flow are, in this case (m = 0), localized also near the inner boundary. As $v_1(1 + \xi)/v_0\xi \to 1$, the dependence of $\sqrt{T_*}$ on this parameter along curves 4 and 5 becomes asymptotic. The results of the calculations agree with the data in [3] with good accuracy.

In order to study the stability of the flow completely with increasing rotation of the outer cylinder $v_1(1 + \xi)/v_0\xi \geqslant 1$, it is sufficient to follow the change in the local maxima of the most critical characteristic values.

Figure 4 shows the change in the maxima of $Y(\alpha)$ for the three most critical azimuthal modes (m = 0, m = 1, and m = 2) with increasing v_1/v_0 ($\xi = 1$). Curves 0–2 correspond to m = 0, m = 1, and m = 2. Maxima of $Y(\alpha)$ for these perturbations with critical Taylor numbers and a definite, for each type of perturbation, relative rotational velocity of the outer cylinder v_1/v_0 have a damping decrement Y = 0. As the parameter v_1/v_0 increases with constant Taylor number $\sqrt{T} = \sqrt{T_*}$, $\max Y(\alpha)$ moves sharply into the region of negative Y values.

Along curves 0 and 1, max $Y(\alpha)$ varies for perturbations with $m = 0$ and $m = 1$, respectively ($\sqrt{T} = 320$ for $m = 0$, $\sqrt{T} = 665$ for $m = 1$). The wave numbers, for which $Y(\alpha)$ attains a maximum value, increase with increasing v_1/v_0.

The behavior of curve 2 reflects the change in the maximum of $Y(\alpha)$ for perturbations with $m = 2$ ($\sqrt{T} = 890$). For such perturbations, a certain nonmonotonic behavior is observed in the behavior of $Y(\alpha)$ for $v_1/v_0 > 0.5$, but this has no effect on the general picture of the instability, since the corresponding max $Y(\alpha)$ lies in the deeply negative region up to $v_1/v_0 = 2$. Along curve 2, and the wave number is observed to increase, i.e., max $Y(\alpha)$ shifts with increasing v_1/v_0 toward the short wavelength region.

Thus, a numerical analysis of the flow with free inner boundaries has shown that the presence of a free cylindrical surface leads to a decrease in the stability of the rotating flow for wide and narrow gaps.

The rotation of the outer cylinder has a stabilizing effect on the stability of the flow. This effect is also observed in the case of a flow between coaxial cylinders for the types of perturbations examined above. The critical Taylor numbers are smallest for axisymmetric perturbations with arbitrary values of ξ, $v_1(1+\xi)/v_0\xi \leqslant 1$ and small Reynolds numbers.

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